Chapter 1

Sets, Boolean algebras, exponential formulae and equations

This chapter sets out the fundamental concepts of set theory and Boolean algebra that will be used to construct a formal theory of reflexive games.

1.1. Sets

A *set* is a collection of distinguishable elements of any nature. An abstract object that does not contain any element is also called a set - an *empty set*. Let us consider a non-empty set that we call *universal* and designate as 1. The set of all subsets of the universal set 1, including the empty set, 0, will be designated as M. We assume that every set includes itself as a subset. A set containing the elements $\alpha, \beta, \gamma, \ldots$ is denoted as $\{\alpha, \beta, \gamma, \ldots\}$.

The expression $A \supseteq B$, where $A \in M$ and $B \in M$, means that B is a subset of A; B may be the set A itself. The expression $A \supset B$ means that B is a proper subset of A, $B \neq A$. We will designate the union of two sets as $+$, and their intersection as $\cdot$.

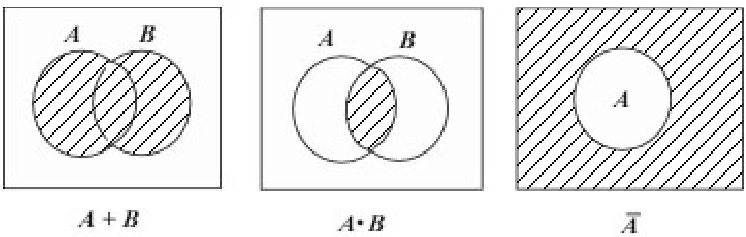

Fig.1.1.1. Venn diagrams.
The square in the figure designates the universal set;
its shaded part is the result of the given operation.

A line above a letter designates the unary operation of complement to the given set, that is, $\overline{A}$ is the set of elements of the universal set that are not included in A. To simplify working with sets we will use Venn diagrams. Figure 1.1.1 shows examples of union $A+B$, intersection $A \cdot B$, and complement $\overline{A}$.

To simplify notation we will write $B \cdot C$ as BC. We will interpret the expression $A+BC$ to mean that the intersection of B and C is first found and then combined with A. The following correlations hold for subsets of set M:

1. $A + A = A$
2. $A\,A = A$
3. $A + B = B + A$
4. $A\,B = B\,A$
5. $A + (B + C) = (A + B) + C$
6. $A\,(B\,C) = (A\,B)\,C$
7. $A\,(B + C) = AB + AC$
8. $A + BC = (A+B)\,(A+C)$
9. $A + B = \overline{\overline{A}\,\overline{B}}$
10. $A + 0 = A$
11. $A + 1 = 1$
12. $\overline{\overline{A}} = A$
13. $A + \overline{A} = 1$
14. $\overline{1} = 0$

1.2. Boolean algebras

Correlations 1-14 above coincide with axioms of Boolean algebra. Therefore, set M with the operations $+$, $\cdot$, $\overline{}$ and the relation $\supseteq$ can be regarded as a Boolean algebra. If the universal set consists of one element, α, then the Boolean algebra consists of two elements: 1 – the universal set containing element α - and the empty set { }. If the universal set consists of two elements, α and β, then the Boolean algebra consists of four elements:

$$1=\{\alpha, \beta\}, \{\alpha\}, \{\beta\}, 0=\{\ \}.$$

In general, if a universal set consists of k elements, the corresponding Boolean algebra consists of 2^k elements (power set).

It is convenient to represent Boolean algebras as lattices: edges correspond to relations of the form $A \supset B$, where B is an element located below A. Examples are given in Figures 1.2.1 and 1.2.2.

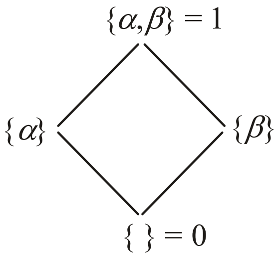

Fig.1.2.1. Boolean lattice corresponding to universal set of two elements, α and β

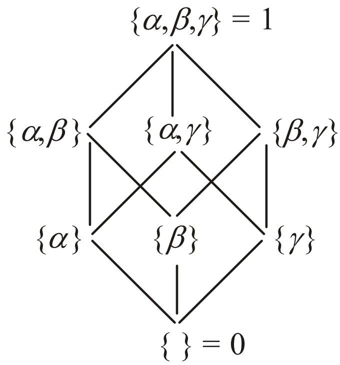

Fig.1.2.2. Boolean lattice corresponding to universal set of three elements, α, β and γ

From now on, in speaking of the set M, i.e., the set of all subsets of the universal set 1, we will keep in mind that M is a Boolean algebra with operations $+$, $\cdot$, $\bar{\ }$ and relation $\supseteq$. It is possible to define functions on the set M that map groups of elements from M onto elements from M. For example,

$$f(a,b,c) = a + bc,$$

where $a,b,c \in M$. This function maps any three elements a,b,c onto $a + bc$.

1.3. Exponential formulae

The following equation plays an important role in our considerations:

$$\Phi(a,b) = a + \bar{b} \,. \tag{1.3.1}$$

We will write it conventionally in exponential form

$$\Phi(a,b) = a^b \tag{1.3.2}$$

and accept the convention for multi-leveled exponents

$$a^{b^c} = a^{\left(b^c\right)}. \tag{1.3.3}$$

The following correlations hold:

1. $a^b a^c = a^{b+c}$
2. $\left(a^b\right)^c = a^{bc}$
3. $(ab)^c = a^c b^c$
3. $a^b + a^c = a^{bc}$
5. $(a+b)^c = a^c + b^c$
6. $(a+b)^c = a^c + b$
7. $a^c + b = a + b^c$
8. $a^{a+b} + b^{a+b} = 1$
9. $a^a = 1$
10. $a^{ab} = 1$
11. $(a+b)^a = 1$
12. $a^b + b^a = 1$
13. $a^0 = 1$
14. $1^a = 1$
15. $a^1 = a$
16. $a^{\overline{a}} = a$
17. $0^a = \overline{a}$

Every exponential expression can be transformed into linear notation. For example,

$$a^{b^{c+d}} = a^{b+\overline{c+d}} = a^{b+\overline{c}\,\overline{d}} = a + \overline{b + \overline{c}\,\overline{d}} = a + \overline{b}\;\overline{\overline{c}\,\overline{d}} = a + \overline{b}(c+d) \quad .$$

In many cases, it is not necessary to transform an expression into linear form; it may be easier to substitute the values of variables directly to an exponential equation. Let the formula

$$a^{b^{cd}},$$

be defined on set M of the universal set $\{\alpha,\beta,\gamma\}$. Set M consists of eight elements:

$$1 = \{\alpha, \beta, \gamma\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\ \} = 0.$$

Let $a = \{\alpha\}$, $b = \{\alpha, \beta\}$, $c = \{\beta, \gamma\}$, $d = 1$. By substituting these values into the exponential form we obtain

$$\{\alpha\}^{\{\alpha,\beta\}^{\{\beta,\gamma\}1}} = \{\alpha\}^{\{\alpha,\beta\}+\overline{\{\beta,\gamma\}}} = \{\alpha\}^{\{\alpha,\beta\}+\{\alpha\}} =$$
$$= \{\alpha\}^{\{\alpha,\beta\}} = \{\alpha\} + \overline{\{\alpha,\beta\}} = \{\alpha\} + \{\gamma\} = \{\alpha,\gamma\}.$$

Computations of complex exponential formulae and universal sets with large numbers of elements can be done in a similar way.

1.4. Equations

Consider function

$$y = Ax + B\overline{x}\ , \tag{1.4.1}$$

where x, A, $B \in M$; A and B do not depend on x.

Statement 1.4.1. Equation

$$Ax + B\overline{x} = x \tag{1.4.2}$$

has a solution if and only if

$$A \supseteq B\,. \tag{1.4.3}$$

Proof. Let the equation have a solution. Then $Ax = x$ and $B\overline{x} = 0$. It follows from the first equality that $A \supseteq x$, and from the second that $x \supseteq B$, because if the complement to x does not intersect with B, then set B is within x. Therefore, $A \supseteq B$. Now let $A \supseteq B$. Choose x such that $A \supseteq x \supseteq B$. It is clear that $Ax = x$ and $B\overline{x} = 0$, since the intersection of B with the complement to x is empty. Therefore, x is a solution of (1.4.2) □

It follows from the preceding proof that every x from the interval $A \supseteq x \supseteq B$ is a solution of equation (1.4.2).

Examples. Consider the following equation:

$$Px + R = x. \tag{1.4.4}$$

Let us find whether (1.4.4) has a solution. To do so, we must represent it as (1.4.2). Since $x + \overline{x} = 1$, we can write (1.4.4) as

$$Px + R(x + \overline{x}) = x$$

and transform it into

$$(P + R)x + R\overline{x} = x\ .$$

In this equation, $P + R$ plays the role of A, and R plays the role of B, that is,

$$(P + R) \supseteq R\ .$$

Thus, (1.4.4) has at least one solution. It belongs to the interval

$$(P + R) \supseteq x \supseteq R\ .$$

Consider another equation:

$$P + R\overline{x} = x\ , \tag{1.4.5}$$

where $P \subset (P + R)$.

We transform its left side:

$$P(x + \overline{x}) + R\overline{x} = Px + (P + R)\overline{x}\ .$$

In this case,

$$A = P,\ \ B = P + R\ .$$

We see that $A \subset B$, i.e., condition (1.4.3) is not met, which means that equation (1.4.5) does not have a solution.

Finally, consider an equation that takes its values from set M of the universal set $\{\alpha, \beta, \gamma\}$:

$$\{\alpha,\beta\}x + \{\beta,\gamma\}\overline{x} = x \quad , \tag{1.4.6}$$

$$A = \{\alpha,\beta\},\ B = \{\beta,\gamma\} \ .$$

In this case, condition (1.4.3) is not met because B is not a subset of A; thus, equation (1.4.6) does not have a solution.

Equations can be written in an exponential form, as well. For example,

$$a^{b+c^{x}} = x \ . \tag{1.4.7}$$

We transform the left part:

$$a^{b+c^{x}} = a^{b+c+\overline{x}} = a + \overline{b}\,\overline{c}x \ .$$

Now the equation has the form

$$a + \overline{b}\,\overline{c}x = x \ .$$

Let us represent the left side as $Ax + B\overline{x}$:

$$a + \overline{b}\,\overline{c}x = a(x + \overline{x}) + \overline{b}\,\overline{c}x = (a + \overline{b}\,\overline{c})x + a\overline{x} \ ,$$

$$A = a + \overline{b}\,\overline{c},\ B = a \ .$$

We see that $A \supseteq B$, and the solutions of (1.4.7) are given by the inequalities

$$(a + \overline{b}\,\overline{c}) \supseteq x \supseteq a \ .$$