Chapter 2

Complete graphs with edges of two types

We assume that any two subjects within a group are either in a relationship of cooperation or in confrontation. We can represent a group as a graph, whose nodes correspond to subjects and whose edges correspond to the relations between them. Graphs in which every two nodes are linked are called *complete*. The edge connecting two nodes, a and b, is designated (a,b). Edges (a,b) and (b,a) are equivalent. A graph is called *elementary* if it consists of one node. The set of all edges of a non-elementary graph can be divided into two nonintersecting subsets, one of which may be empty. We will call these subsets *relations* R and $\overline{R}$. One of them is interpreted as cooperation, and the other as conflict. If $(a,b) \in R$, we say that a and b are linked by R, and write aRb. If $(a,b) \in \overline{R}$, then a and b are linked by $\overline{R}$, which is written $a\overline{R}b$. All definitions for R hold for $\overline{R}$. If two nodes, a and b, can be linked by a sequence of R-edges, we say that a and b are linked in R. If every node of graph A is linked with every node of graph B in R, we will write it ARB. In this case, we say that A and B are in relation R. Thus, relation R between nodes is transferred to a relation between graphs. Expression aRb will also designate the relation between elementary graphs consisting of nodes a and b, respectively. If graph G consists of subgraphs that are mutually in relation R, we say that G is *divided* into these subgraphs; we will write this down as $G = A_1RA_2R...RA_n$, where $A_1,A_2,\dots,A_n$ are subgraphs. Expression $B \subseteq A$ means that graph B is a subgraph of A: B's set of nodes is a subset of A's nodes, its edges are induced by A (if B is not an elementary graph), that is, every edge in B linking two of its nodes a and b

coincides with an edge in A linking these nodes. Expression $B \subset A$ means that B is a subgraph of A, but does not coincide with A. If $B \subseteq A$ and $C \subseteq A$, then $D = B \cup C$ means that D is a union of B's and C's sets of nodes with edges (if they exist) induced by A, and $E = B \cap \mathrm{C}$ means that E is an intersection of B's and C's sets of nodes (if the intersection is not empty), with edges (if they exist) induced by A. Expression $a \in A$ means that a is a node of graph A, and $A_{(k)}$ means that graph A consists of k nodes. Expression G–A denotes graph G, from which all A's nodes are taken away. Expression $<a,b,\ldots>$ denotes a graph with nodes a, b, ... Further on we will consider complete and elementary graphs. The proofs of statements formulated in this Chapter and in the next one are given in Appendix 10 to the Lefebvre's book (2001).

2.1. Basic definitions

Definition 1. Graph G is *stratified* in R, if it can be represented as $G = ARB$. Graphs A and B are called *strata* of graph G in R.

Definition 2. Graph G is *totally stratified*, if each of its non-elementary subgraphs is stratified either in R or in $\overline{R}$.

Definition 3. If A is a stratum of G in R, and if A is not stratified in R, A is called a *minimal stratum* of G in R.

2.2. Theorem on total stratification

A graph consisting of four nodes linked both in R and in $\overline{R}$, will be denoted $S_{(4)}$. An example of such a graph is given in Fig. 2.2.1. Solid lines depict relation R, and broken ones depict relation $\overline{R}$. The graph is linked both by solid lines and broken ones.

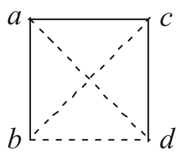

Fig. 2.2.1. Graph $S_{(4)}$.

Theorem on total stratification. Graph G is totally stratified if and only if its subgraphs contain no graph $S_{(4)}$. (A proof is given in Batchelder, Lefebvre, 1982; see also Lefebvre, 2001.)

Let us look at the graphs in Figures 2.2.2 and 2.2.3.

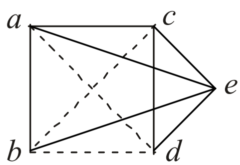

Fig. 2.2.2. A stratified graph that is not totally stratified.

The graph in Fig. 2.2.2 is stratified because it can be represented as $\langle a,b,c,d\rangle R\langle e\rangle$. But this graph contains the subgraph $\langle a,b,c,d\rangle$, which is $S_{(4)}$. Therefore graph $\langle a,b,c,d,e\rangle$ is not totally stratified.

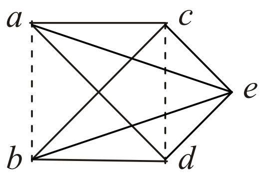

Fig. 2.2.3. A totally stratified graph

None of the four-node subgraphs of the graph in Fig.2.2.3 is

$S_{(4)}$. So, it follows from the theorem on total stratification that the graph in Fig.2.2.3 is totally stratified. Therefore, to determine whether a graph is totally stratified, one needs to find out if there is subgraph $S_{(4)}$ among its subgraphs. If there is no $S_{(4)}$, the graph is totally stratified; if there is a subgraph $S_{(4)}$, it is not. Note that graphs with two or three nodes are stratified.

Consider a graph that is not totally stratified. If we begin taking away its nodes one by one, together with their adjacent edges, in a certain number of steps we will reach a totally stratified graph, since a three-node graph is always totally stratified.