Chapter 3
Decomposable graphs, polynomials, and diagonal forms

In this chapter we present a formal apparatus for analysis of subjects and groups.

3.1. Theorem on decomposition.

The proofs of the following statements are given in Appendix 10 Lefebvre, 2001.
Statement 3.1.1. A graph cannot be stratifiable both in R and in $\overline{R}$ at the same time.
Statement 3.1.2. If a graph is stratifiable in R, its division into minimal strata in R is unique to within strata's numeration.

These statements underlie the procedure for decomposition of a stratifiable graph; we call it the D-procedure.

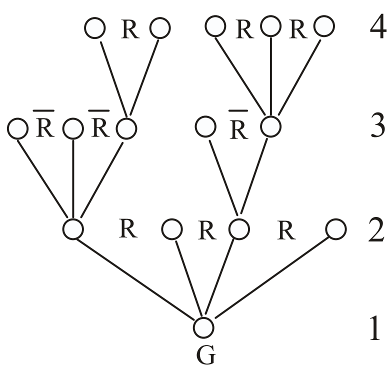

Fig. 3.1.1. Decomposition tree for graph G;
tiers numbers are given on the right.

It consists of progressive division of a graph into minimal strata. Each minimal stratum belongs to a certain numbered level of division of tiers. We investigate each minimal stratum that is in relation R with other minimal strata to check whether it is stratifiable in relation $\overline{R}$. (According to the definition of a minimal stratum, it is not stratifiable in R.) If it is not stratifiable in $\overline{R}$, the investigation ends; if it is stratifiable in $\overline{R}$, we divide it into minimal strata in $\overline{R}$, which strata belong to the next tier. This procedure generates trees of the type given in Fig. 3.1.1. Relations R and $\overline{R}$ alternate: R - on the second tier, $\overline{R}$ - on the third, R - on the fourth, etc.

Every circle in Fig.3.1.1 corresponds to a subgraph of graph G; symbols R and $\overline{R}$ depict relations between minimal strata. If a circle is an *end* with no originating branches, it corresponds either to an elementary graph of one node or to a non-stratifiable graph. By Statement 3.1.2, the decomposition tree of a stratifiable graph is unique to within the order of branches originating in circles.

We call a graph *decomposable*, if it is not elementary and every ending circle in its decomposition tree is an elementary graph.

Theorem of decomposition. Graph G is decomposable if and only if it is totally stratifiable. (Batchelder, Lefebvre, 1982; see also Lefebvre, 2001).

It follows from this statement that a graph is decomposable if and only if there is no $S_{(4)}$ among its subgraphs (by the theorem of total stratification).

3.2. Graphs and grammatical trees

Decomposable graphs can be represented in analytical notation to facilitate working with them. Here is the procedure for translating a decomposable graph into analytical notation. First, we construct a decomposition tree. Then, we construct a *grammatical tree*, isomorphic to the decomposition tree. Symbols R and $\overline{R}$ appear in the same places in the grammatical tree as in the decomposition tree. Ends of the branches in the grammatical tree which do not have originating branches are denoted by the letters that correspond to the graph's nodes. Ramifications are denoted by other letters. Letters at the ends of branches are called *end* letters, and others are called *intermediate* letters. Each intermediate letter designates a group of symbols (letters, signs R and $\overline{R}$, parentheses) located immediately above it and can be replaced by this group taken in parentheses. The end result is a *word*, taken to be the analytical equivalent of the graph.

Consider graph in Fig. 3.2.1.

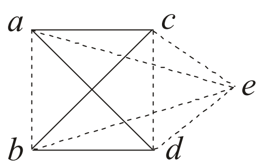

Fig. 3.2.1. Decomposable graph

Solid lines in Fig. 3.2.1. correspond to relation R, and dotted ones to $\overline{R}$. This graph is decomposable because it does not contain the subgraph $S_{(4)}$. Its decomposition tree is given in Fig. 3.2.2, and a grammatical tree for the same graph in Fig. 3.2.3.

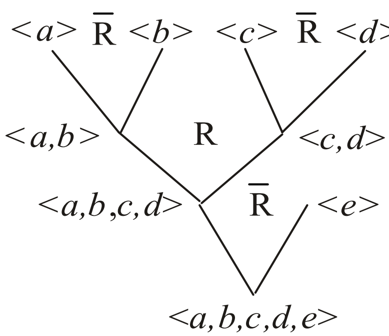

Fig. 3.2.2. Decomposition tree of the graph in Fig. 3.2.1

Letters a, b, c, d and e are end letters, and A_1, A_2, A_3 and A_4 intermediate. A_3 designates expression $a\overline{R}b$, A_4 designates $c\overline{R}d$. We put these expressions in parentheses and substitute them for A_3 and A_4.

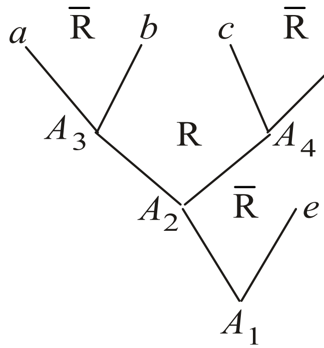

Рис. 3.2.3. Grammatical tree of the graph given in Fig. 3.2.1

Letter A_2 designates $(a\overline{R}b)R(c\overline{R}d)$. We put this expression in parentheses and substitute it for A_2. Finally, we put the expression $((a\overline{R}b)R(c\overline{R}d))\overline{R}e$ in parentheses to replace A_1. The resulting grammatical tree appears as shown in Fig. 3.2.4.

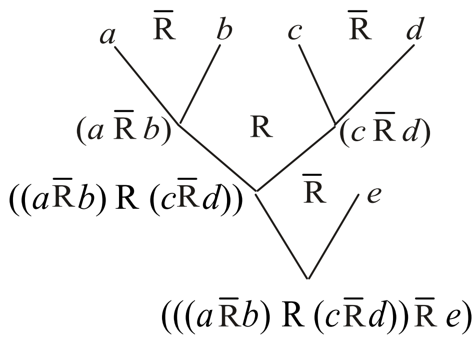

Fig. 3.2.4. Grammatical tree after replacement of the letters A_1, A_2, A_3 and A_4 by the expressions they designate; analytical notation at the bottom corresponds to the graph in Fig. 3.2.1

Any decomposable graph can be represented by a similar tree. Each node in the tree corresponds to the analytical notation of some subgraph.

3.3. Polynomials and diagonal forms

Next, we will consider letters in the analytical notation of a graph as variables defined on the set M of all subsets of a universal set and symbols R and $\overline{R}$ as the operations of intersection ($\cdot$) and union (+), defined on the same set. Further, we will substitute $\cdot$ for one of the symbols, R or $\overline{R}$, and + for the other. We will call $\cdot$ *multiplication* and + *addition*. The analytical notation of a graph with the above interpretation is called *polynomial*. A polynomial consisting of one letter is called *elementary* and corresponds to an elementary graph. After substitution $\cdot$ for R and + for $\overline{R}$, the analytical notation in the bottom of Fig. 3.2.4 becomes the following polynomial:

$$(((a + b) \cdot (c + d)) + e). \qquad (3.3.1)$$

Polynomials can be written within brackets. Let us agree that instead of $[(A)]$ one can write $[A]$ and instead of $(A \cdot B)$ write $A \cdot B$ or AB. Then, polynomial (3.3.1) appears as follow:

$$[(a + b)\,(c + d) + e], \qquad (3.3.2)$$

and the grammatical tree in Fig. 3.3.4 appears as a tree of polynomials (Fig.3.3.1):

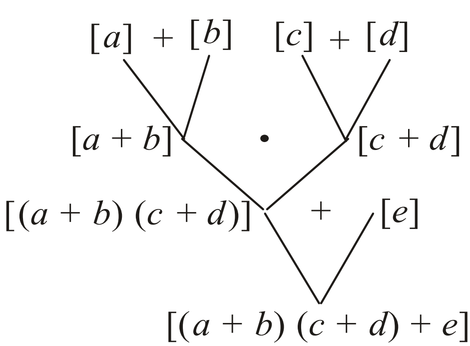

Fig. 3.3.1. Tree of polynomials

Now let us eliminate the lines. For each non-elementary polynomial, we will write the polynomials located directly above it, and to the right. Thus, we obtain a treelike object called a *diagonal form*. For the tree in Fig. 3.3.1, the diagonal form appears as follows:

$$\begin{array}{llllll} & & & [a] + [b] & [c] + [d] & \\ & & [a + b] & & [c + d] & \\ & [(a + b)\ (c + d)] & & & & + [e] \\ [(a + b)\ (c + d) + e] & & & & & . \end{array}$$

Expression $[a]$, where a is an elementary polynomial, will be called an elementary diagonal form. *Non-elementary diagonal form is interpreted an exponential formula* (see Chapter 1), where the exponent P^W corresponds to the function $\Phi = P + \overline{W}$. *Parentheses and brackets are considered equivalent in computation.*

For the practical purpose of finding a polynomial corresponding to a simple decomposable tree, one may omit the step of explicitly creating a grammatical tree. Also, outside of diagonal forms brackets are not needed.

Consider a graph of three nodes (Fig.3.3.2):

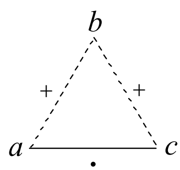

Fig. 3.3.2. A decomposable graph

This graph can be divided into two subgraphs. The first one, consisting of one node, b, is an elementary graph; the second one consists of the two nodes a and c. The node of the first subgraph, b, is in the same relation + with nodes a and c of the second subgraph. This means that the first and second subgraphs are in relation +. We cannot decompose these two subgraphs to smaller subgraphs in relation +; thus, they are minimal strata in this relation. The second subgraph can be decomposed to two subgraphs consisting of the nodes *a* and *c* in relation $\cdot$. The resulting polynomial is

$$b + ac.$$

Fig. 3.3.3 shows all graphs of the three nodes *a*, *b*, *c* and their corresponding polynomials:

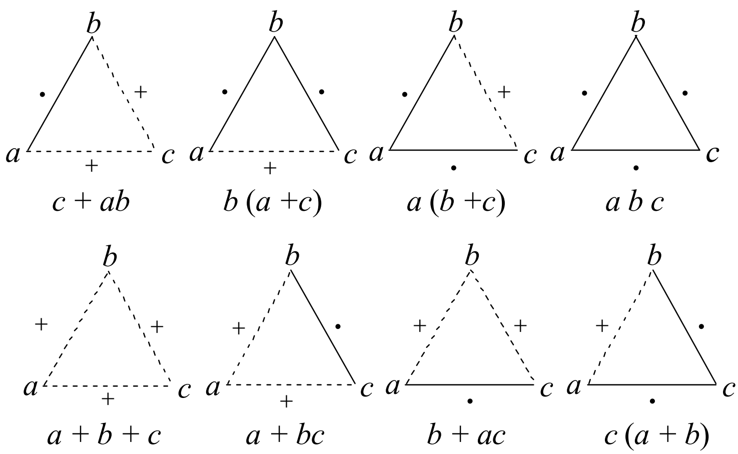

Fig. 3.3.3. Graphs of three nodes and their corresponding polynomials

Polynomials for decomposable graphs with a greater number of nodes can be found in a similar way. Consider, for example, a decomposable graph of four nodes (Fig.3.3.4):

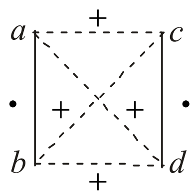

Fig. 3.3.4. A decomposable graph

This graph decomposes into two subgraphs with nodes a, b and c, d. The nodes of the first subgraph are connected with the nodes of the second subgraph by +; thus, the subgraphs are in the same relation. Neither subgraph decomposes to smaller subgraph in relation +; that is, they are minimal strata in +. But each subgraph can be decomposed into elementary subgraphs in relation $\cdot$. Therefore, the graph in Fig. 3.3.4 corresponds to the polynomial

$$ab + cd.$$

Note that a polynomial must correspond to the graph's analytical notation, and we must not perform any transformation of the polynomial, except the commutative, before a diagonal form is constructed.

Let us describe the procedure for construction of diagonal form when a polynomial is known. We will call a polynomial minimal in $\cdot$ if it cannot be represented as a product $A \cdot B$, and minimal in + if it cannot be represented as a sum $A+B$.

1. Enclose the polynomial into brackets.

2. If it is elementary, the procedure ends.

3. If the polynomial can be represented as a product, divide it into multiplier-polynomials minimal in $\cdot$ and write them above and to the right one after another, each enclosed in square brackets.

4. If the polynomial can be represented as a sum, divide it into summand-polynomials minimal in +, and write them above and to the right, each in square brackets and connected by +.

5. For every polynomial written above and to the right, repeat the procedure starting from 2.

6. The procedure ends when every non-elementary polynomial has diagonal elements.

Consider the graph in Fig.3.3.2. It corresponds to polynomial $b + ac$. Let us construct its diagonal form. First, put the polynomial in brackets:

$$[b + ac].$$

The initial polynomial is a sum of two polynomials minimal in $+$: b and ac. Put them into brackets, connected by $+$, and write them above and to the right of the initial polynomial:

$$\begin{array}{lll} & [b] + [ac] & \\ [b + ac] & & . \end{array}$$

Polynomial $[b]$ is elementary and cannot be decomposed further, but polynomial $[ac]$ is not elementary, it is a product of two polynomials, a and c. Put them into brackets and write one after another above and to the right of the polynomial $[ac]$:

$$\begin{array}{llll} & & [a]\,[c] & \\ & [b] + [ac] & & \\ [b + ac] & & & . \end{array} \qquad (3.3.3)$$

The construction of the diagonal form is complete.

Each diagonal form corresponds to a function composed of Boolean operations $+$, $\cdot$, $\overline{\ \ }$, and can be represented as

$$Aa + B\overline{a} \qquad (3.3.4)$$

on any of its arguments a, where A and B do not depend on a.

Further on we will use two interpretations of diagonal form. On the one hand, it is a description of the subject with the inner domain; on the other, it is an exponential formula representing a function for computation. Therefore, a diagonal form plays the role of a picture and of a formula.